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S-duality in Vafa-Witten theory for non-simply laced gauge groups

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Abstract: Vafa-Witten theory is a twisted $N = 4$ supersymmetric gauge theory whose partition functions are the generating functions of the Euler number of instanton moduli spaces. In this paper, we recall quantum gauge theory with discrete electric and magnetic fluxes and review the main results of Vafa-Witten theory when the gauge group is simply laced. Based on the transformations of theta functions and their appearance in the blow-up formulae, we propose explicit transformations of the partition functions under the Hecke group when the gauge group is non-simply laced. We provide various evidences and consistency checks.

Keywords: Topological Field Theories, Duality in Gauge Field Theories, Differential and Algebraic Geometry, Supersymmetric gauge theory.

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Contents

1. Introduction

One of the most fascinating conjectures in quantum field theory is the Montonen-Olive duality [1] between electricity and magnetism as the gauge group is exchanged with its dual [2]. It was realised subsequently that this duality is more likely to hold in supersymmetric gauge theories [3, 4]. If such a theory can be twisted so that part of the supersymmetry remains on a curved four-manifold, the theory becomes topological in the sense that the partition function and observables are topological or smooth invariants. In such a case, duality can both be tested by known mathematical results and predict new ones. A celebrated example is the twisted $N = 2$ supersymmetric gauge theory [5], in which the duality of low energy descriptions [6] yields a relation between Donaldson and Seiberg-Witten invariants [7]. The $N = 4$ supersymmetric gauge theory is believed to have exact electric-magnetic duality, or S-duality [4]. It has three inequivalent twists [8, 9]. One of the two twists in [8] is the Vafa-Witten theory [10], whose partition functions are the generating functions of the Euler number of instanton moduli spaces. These partition functions depend on the discrete fluxes of 't Hooft [11] and they transform under the modular group $\mathrm{SL}(2, \mathbb{Z})$ for simply laced gauge groups. This sharpened S-duality conjecture [10] impose stringent constraints on the Euler number of moduli spaces and has been a fruitful source of development in both topology and S-duality. Recently, there has been much interest in the third twist [9] for its relation to the geometric Langlands programme [12 − 15].

In this paper, we revisit the Vafa-Witten theory with an emphasis on the roles of Langlands duals. While much progress has been made when the gauge group is simply laced, both in physics [16 − 20] and in mathematics [21 − 25], there has been almost no attempt in studying the Vafa-Witten theory in the non-simply laced case (see however [26]).

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The latter differs from the simply laced case in several crucial ways. First, as the gauge group is distinct from its dual, we need to consider simultaneously two sets of discrete fluxes and hence two sets of partition functions. Second, the $\mathbb{Z}_2$-duality of electricity and magnetism is not part of the modular group, but of the Hecke group [27 − 29], which has different relations on the generators. Based on the transformations of theta functions [30] and their appearance in the blow-up formulae [26], we propose explicit transformations of the partition functions with various discrete fluxes under the Hecke group. This would be the counterpart, when the gauge group is non-simply laced, of the sharpened S-duality conjecture of Vafa and Witten [10].

The organisation of the paper is as follows. In section 2, we review 't Hooft's discrete electric and magnetic fluxes [11] and their role in canonical and path integral quantisation. Given an arbitrary simple gauge group, we choose a subset of permitted discrete fluxes so that under S-duality, the discrete electric and magnetic fluxes are interchanged. In section 3, we consider Vafa-Witten theory for simply laced gauge groups. The sharpened S-duality conjecture [10] specifies how the partition functions with various discrete fluxes transform under the modular group. From this and the above selection of discrete fluxes for arbitrary gauge groups, we deduce the usual $\mathbb{Z}_2$-duality which exchanges the gauge group and its Langlands dual. To compare with the non-simply laced case, we summarise the relevant mathematical results, especially the blow-up formulae [22, 10, 24, 26], in which the universal factors contain theta functions constructed from the coroot lattice [26] and the Dedekind eta function. In section 4, we study Vafa-Witten theory for non-simply laced gauge groups. There are two sets of partition functions which transform under the Hecke group. To find the explicit representation, we start from the theta functions whose transformations under the Hecke group are known [30]. Their appearance in the blow-up formulae is then used to determine how the generators of the Hecke group act on the partition functions. Our result contains new phase factors which are different from those proposed in [10]. We then explain the consequences on the blow-up formulae and compactification of moduli spaces. Finally, we check that the action of the generators indeed defines a representation of the Hecke group. In section 5, we discuss some possible future directions from this work. We collect in appendix A some facts on Lie algebras, Lie groups, their Langlands duals, invariant bilinear forms and Coxeter, dual Coxeter numbers. In appendix B, we review the geometry of fractional instanton numbers [10] in the presence of discrete fluxes for arbitrary gauge groups. We also mention some congruence properties of the signature of four-manifolds and the dimension of instanton moduli spaces.

2. Gauge theory with discrete electric and magnetic fluxes

We consider gauge theories on a Riemannian four-manifold X with a compact, simple gauge group G. The fields in such a theory are the gauge potential A, which is a connection on a principal G-bundle P and the matter fields, denoted by ψ collectively, which are sections of various associated bundles of P. The action is

$$S[A,\psi] = \int_X \left[\frac{1}{e^2}(F \mid *F) + \frac{\sqrt{-1}\,\theta}{8\pi^2}(F \mid F) + \cdots \right]$$

$$= \int_X \frac{\sqrt{-1}}{4\pi} \left[\bar{\tau}\,(F^+|F^+) + \tau\,(F^-|F^-)\right] + \cdots, \tag{2.1}$$

where the terms with ψ and the coupling to A are omitted. Here, the pairing on the curvature 2-form is given by the bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ explained in appendix A and the wedge product on forms. The gauge coupling constant e and the θ angle combine as a complex coupling $\tau = \theta/2\pi + 4\pi\sqrt{-1}/e^2$ in the upper-half plane.

In quantum theory, we integrate over A and ψ. When G is simply connected, the instanton numbers are summed over in the path integral in order to be compatible with the Hamiltonian formalism. When G is not simply connected, there are additional characteristic classes and the instanton numbers are no longer integers [10]. If $G = G_{\mathrm{ad}}$, we have $w_2(P) \in H^2(X, \mathcal{Z})$, where $\mathcal{Z}$ is the centre of the universal covering group $\tilde{G}$, and an instanton number k satisfying (B.4). Fixing $w_2(P) = v$, the partition function is

$$Z_{X,v}(\tau) = \sum_{k\in\mathbb{Z}-\frac{1}{2}(v|v)} \frac{1}{\mathrm{vol}(\mathcal{G}_{k,v})} \int_{\substack{k(P)=k\\ w_2(P)=v}} DA\, D\psi\; e^{-S[A,\psi]}, \tag{2.2}$$

where $\mathcal{G}_{k,v}$ is the group of gauge transformations on a bundle P with the prescribed topology. Henceforth, we omit the subscript X in $Z_{X,v}(\tau)$ unless confusion occurs. Because of the fractional instanton numbers, we have

$$Z_v(\tau+1) = e^{-\pi\sqrt{-1}\,(v|v)}\, Z_v(\tau). \tag{2.3}$$

To relate to canonical quantisation, we consider the case $X = S^1 \times Y$, where S^1 is the time direction and Y is a spatial three-manifold. We write $v = (a, m)$ according to the decomposition [12]

$$H^2(X, \mathcal{Z}) \cong H^1(Y, \mathcal{Z}) \oplus H^2(Y, \mathcal{Z}). \tag{2.4}$$

Following [11], m is called a discrete magnetic flux and an element $e \in H^1(Y, \mathcal{Z})^\wedge = \mathrm{Hom}(H^1(Y, \mathcal{Z}), U(1))$ is called a discrete electric flux. For each pair (e, m), the partition function

$$Z_{e,m}(\tau) = \sum_{a\in H^1(Y,\mathcal{Z})} e(a)\, Z_{v=(a,m)}(\tau) \tag{2.5}$$

corresponds to some Hilbert space $\mathcal{H}_{e,m}$ in the Hamiltonian formalism.

For $G = \tilde{G}$ and $G = G_{\mathrm{ad}}$, the Hilbert spaces are, respectively, $\mathcal{H}_{\tilde{G}} = \oplus_{e\in H^1(Y,\mathcal{Z})^\wedge} \mathcal{H}_{e,0}$ and $\mathcal{H}_{G_{\mathrm{ad}}} = \oplus_{m\in H^2(X,\mathcal{Z})} \mathcal{H}_{0,m}$ [12]. The corresponding partition functions are

$$Z_{\tilde{G}}(\tau) = \sum_{e\in H^1(Y,\mathcal{Z})^\wedge} Z_{e,m=0}(\tau) = |\mathcal{Z}|^{b_1(Y)}\, Z_{v=0}(\tau), \tag{2.6}$$

$$Z_{G_{\mathrm{ad}}}(\tau) = \sum_{m\in H^2(X,\mathcal{Z})} Z_{e=0,m}(\tau) = \sum_{\substack{a\in H^1(Y,\mathcal{Z})\\ m\in H^2(Y,\mathcal{Z})}} Z_{v=(a,m)}(\tau). \tag{2.7}$$

For a general gauge group G with the same Lie algebra $\mathfrak{g}$, we choose the partition function as

$$Z_G(\tau) = \sum_{\substack{e|_{H^1(Y,\pi_1(G))}=1\\ m\in H^2(Y,\pi_1(G))}} Z_{e,m}(\tau). \tag{2.8}$$

In addition to having a Hilbert space

$$\mathcal{H}_G = \bigoplus_{\substack{e|_{H^1(Y,\pi_1(G))}=1 \\ m\in H^2(Y,\pi_1(G))}} \mathcal{H}_{e,m}, \tag{2.9}$$

this prescription has the following two (somewhat related) advantages. First, the partition function can be written as

$$Z_G(\tau) = |Z(G)|^{b_1(Y)} \sum_{\substack{a\in H^1(Y,\pi_1(G)) \\ m\in H^2(Y,\pi_1(G))}} Z_{v=(a,m)}(\tau) = |Z(G)|^{-1+b_1(X)} \sum_{v\in H^2(X,\pi_1(G))} Z_v(\tau), \tag{2.10}$$

an expression which is manifestally relativistic. (See [10] for a derivation of the factor $|Z(G)|^{-1+b_1(X)}$ without the space-time splitting.) Second, the restriction on e in (2.8) is equivalent to $e \in H^1(Y, Z(G))^\wedge$. By Poincaré duality, $H^2(Y,\pi_1({}^LG)) \cong H^1(Y,Z(G))^\wedge$, $H^1(Y,Z({}^LG))^\wedge \cong H^2(Y,\pi_1(G))$. So when G is replaced by its Langlands dual ${}^LG$, the spaces of e and m are exchanged. This makes S-duality possible.

3. Vafa-Witten theory for simply laced gauge groups: modular invariance

In [10], Vafa and Witten studied S-duality in twisted $N = 4$ supersymmetric gauge theory. Such a theory is topological and can be defined on any curved four-manifold X while maintaining part of the supersymmetry. With certain vanishing theorems [10], the partition function captures the Euler number of instanton moduli spaces. Recall that the topology of a G_{ad}-bundle P over a four-manifold X is determined by $w_2(P) \in H^2(X,\mathcal{Z})$ and an instanton number $k(P)$ satisfying (B.4). Fixing $w_2(P) = v$, the partition function is [10]

$$Z_v(\tau) = q^{-s} \sum_{k\in\mathbb{Z}-\frac{1}{2}(v|v)} \chi(\,\overline{\mathcal{M}_{k,v}}\,)\, q^k, \tag{3.1}$$

where $q = e^{2\pi\sqrt{-1}\,\tau}$ and $\mathcal{M}_{k,v} = \mathcal{M}_{k,v}(X)$ is the moduli space of anti-self-dual instantons on X with the prescribed topology. This is the generating function of the Euler number of certain compactification of $\mathcal{M}_{k,v}$. The factor q^{-s} comes from the modification of the action to ensure S-duality in curved space. Consequently, even when $v = 0$ and $k \in \mathbb{Z}$, $Z_v(\tau)$ is not invariant under $\tau \mapsto \tau + 1$. As the theory is topological, s is a linear combination of the Euler number χ and the signature σ of X.

For simplicity, we assume that $H_1(X,\mathcal{Z})$ has no $|\mathcal{Z}|$-torsion as in [10]. Then $H^2(X,\mathcal{Z}) \cong \mathcal{Z}^{b_2}$, where $b_i = b_i(X)$ is the ith Betti number of X. When $\mathfrak{g}$ is simply laced, the sharpened S-duality conjecture of Vafa and Witten [10] is that

$$Z_v\left(-\frac{1}{\tau}\right) = \pm\frac{1}{|\mathcal{Z}|^{b_2/2}}\left(\frac{\tau}{\sqrt{-1}}\right)^{w/2} \sum_{u\in H^2(X,\mathcal{Z})} e^{2\pi\sqrt{-1}\,(v|u)} Z_u(\tau) \tag{3.2}$$

for some modular weight w. We can eliminate the factor $(\tau/\sqrt{-1}\,)^w$ by defining

$$\hat{Z}_v(\tau) = \eta(\tau)^{-w} Z_v(\tau), \tag{3.3}$$

where $\eta(\tau) = q^{1/24} \sum_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. Then the transformations become

$$\begin{aligned} \hat{Z}_v(\tau+1) &= e^{-\pi\sqrt{-1}\,c/12-\pi\sqrt{-1}\,(v|v)}\,\hat{Z}_v(\tau), \\ \hat{Z}_v\left(-\frac{1}{\tau}\right) &= \pm\frac{1}{|\mathcal{Z}|^{b_2/2}} \sum_{u\in H^2(X,\mathcal{Z})} e^{2\pi\sqrt{-1}\,(v|u)}\,\hat{Z}_u(\tau), \end{aligned} \tag{3.4}$$

where $c = 24s + w$ is also a linear combination of χ and σ.

The original Montonen-Olive duality conjecture [1] is a consequence of the sharpened S-duality (3.4). The formula of $\hat{Z}_v(-1/\tau)$ in (3.4) when $v = 0$ already shows the duality between $\tilde{G}$ and ${}^L\tilde{G} = G_{\mathrm{ad}}$ [10, 12]. For any gauge group G with the same Lie algebra $\mathfrak{g}$, the partition function is, according to (2.10),

$$\hat{Z}_G(\tau) = |Z(G)|^{-1+b_1} \sum_{v\in H^2(X,\pi_1(G))} \hat{Z}_v(\tau). \tag{3.5}$$

Thus we have

$$\begin{aligned} \hat{Z}_G\left(-\frac{1}{\tau}\right) &= |Z(G)|^{-1+b_1} \sum_{v\in H^2(X,\pi_1(G))} \pm|\mathcal{Z}|^{-b_2/2} \sum_{u\in H^2(X,\mathcal{Z})} e^{2\pi\sqrt{-1}\,(u|v)}\,\hat{Z}_v(\tau) \\ &= \pm|Z(G)|^{-\chi/2}\,|\pi_1(G)|^{b_2/2} \sum_{v\in H^2(X,\pi_1({}^LG))} \hat{Z}_v(\tau) \\ &= \pm|Z(G)|^{-\chi/2}\,|Z({}^LG)|^{\chi/2}\,\hat{Z}_{{}^LG}(\tau). \end{aligned} \tag{3.6}$$

That is, the quantum theory with gauge group G and coupling $-1/\tau$ is the same as that with gauge group ${}^LG$ and coupling τ. This is the Montonen-Olive duality for a general (simply laced) gauge group.

The constants c, w, s and the sign in (3.4) are fixed by the requirement that (3.4) defines a representation of the modular group and by explicit calculations of examples of four-manifolds. Recall that the modular group $\Gamma = SL(2,\mathbb{Z})$ is generated by $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\colon \tau \mapsto \tau+1$ and $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\colon \tau \mapsto -1/\tau$ satisfying the relations

$$S^2 = (ST)^3 \in Z(\Gamma), \quad S^4 = I. \tag{3.7}$$

The argument in [10] for $\mathrm{SU}(n)$ shows that for any simply laced gauge group, the matrix representations of T and S given by (3.4),

$$\mathcal{T}_{uv} = e^{-\pi\sqrt{-1}\,c/12-\pi\sqrt{-1}\,(v|v)}\,\delta_{uv}, \quad \mathcal{S}_{uv} = \pm|\mathcal{Z}|^{-b_2/2}\,e^{2\pi\sqrt{-1}\,(u|v)}, \tag{3.8}$$

satisfy the relations in (3.7) if

$$c = r_{\mathfrak{g}}\,\chi \bmod 4 \quad \text{and} \quad \pm = (-1)^{r_{\mathfrak{g}}(\chi+\sigma)/4}. \tag{3.9}$$

Here $r_{\mathfrak{g}}(\chi+\sigma)/4 \in \mathbb{Z}$ by (B.8) because the Euler number of $\overline{\mathcal{M}_{k,v}}$ vanishes unless its dimension is even. In fact, it is believed that if the gauge group is simply laced, then [10, 17, 19]

$$s = (r_{\mathfrak{g}}+1)\chi/24, \quad w = -\chi, \quad c = r_{\mathfrak{g}}\,\chi. \tag{3.10}$$

This agrees with, for SU(2) and more generally for SU(n), the calculations of K3 [31, 10, 16, 20], $\mathbb{C}P^2$ [21, 22], $\frac{1}{2}$K3 (rational elliptic surfaces) [16, 23] and rational surfaces [25]. It also agrees with the physics calculation of Kähler surfaces whose canonical divisor is a disjoint union of smooth curves [10, 17]. For other types of simply laced gauge groups, the partition functions have been studied for K3 and $T^4/\mathcal{Z}_2$ [18, 19].

The transformations (3.2), (3.4) with (3.10) is also consistent with the blow-up formulae. Let X be an algebraic surface and $\tilde{X}$, its blow-up at a point. Topologically, $\tilde{X}$ is the connected sum of X and $\overline{\mathbb{C}P^2}$. Thus $H^2(\tilde{X},\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \oplus H^2(\overline{\mathbb{C}P^2},\mathbb{Z})$, where $H^2(\overline{\mathbb{C}P^2},\mathbb{Z})$ has one generator e with the pairing $e^2=-1$. So the discrete fluxes $\tilde{v}$ on $\tilde{X}$ and v on X are related by $\tilde{v}=(v,a\otimes e)$, where $a\in\mathcal{Z}\cong\Lambda^*/\Lambda^\vee$. Blow-up formulae relate the partition functions of the theories on X and those on $\tilde{X}$. The obvious generalisation of the SU(2) case [10] to any simply laced gauge group (see [25, 17] for SU(n)) is

$$\hat{Z}_{\tilde{X},\tilde{v}}(\tau)=\hat{\theta}_a(\tau)\,\hat{Z}_{X,v}(\tau), \tag{3.11}$$

where $\hat{\theta}_a(\tau)=\eta(\tau)^{-r_{\mathfrak{g}}}\theta_a(\tau)$ and

$$\theta_a(\tau)=\sum_{x\in\Lambda^\vee+a} e^{\pi\sqrt{-1}\,(x|x)\tau}. \tag{3.12}$$

Note that $\hat{\theta}_a(\tau)$ ($a\in\mathcal{Z}$) are the level 1 affine characters [32, 33] and transform under the modular group Γ. The representation of Γ on $\{\hat{Z}_{\tilde{X},\tilde{v}}\}$ is the tensor product of those on $\{\hat{Z}_{X,v}\}$ and on $\{\hat{\theta}_a\}$.

Mathematically, (3.11) can be written more explicitly as, for $\tilde{v}=(v,a\otimes e)$,

$$\sum_{k\in\mathbb{Z}-\frac{1}{2}(\tilde{v}|\tilde{v})}\chi(\,\overline{\mathcal{M}_{k,\tilde{v}}(\tilde{X})}\,)\,q^k=\frac{q^{(r_{\mathfrak{g}}+1)/24}}{\eta(\tau)^{r_{\mathfrak{g}}+1}}\,\theta_a(\tau)\sum_{k\in\mathbb{Z}-\frac{1}{2}(v|v)}\chi(\,\overline{\mathcal{M}_{k,v}(X)}\,)\,q^k. \tag{3.13}$$

In fact, the factorisation (3.11) for $G=\mathrm{SU}(2)$ or $SO(3)$ was motivated by Yoshioka's work [22] (when X is projective and $a=0$) and the requirement of S-duality [10]. Li and Qin [24] proved, again when $G=\mathrm{SU}(2)$ or $SO(3)$, that (3.13) holds for any smooth algebraic surface X and $a\in\mathcal{Z}$, if $\overline{\mathcal{M}_{k,v}}$ is the Gieseker compactification of $\mathcal{M}_{k,v}$. The power of the Dedekind eta function in the denominator comes from the boundary components of the moduli spaces included during compactification. Ignoring their contributions, Kapranov [26] showed that

$$\sum_{k\in\mathbb{Z}}\chi(\mathcal{M}_{k,0}(\tilde{X}))\,q^k=\theta_0(\tau)\sum_{k\in\mathbb{Z}}\chi(\mathcal{M}_{k,0}(X))\,q^k \tag{3.14}$$

for any (possibly non-simply laced) gauge group. This provides further support for the appearance of $\theta_a(\tau)$ in the universal factor on the right hand side of (3.13).

4. Vafa-Witten theory for non-simply laced gauge groups: the Hecke group

For non-simply laced gauge groups, duality exchanges the parameter τ with $-1/n_{\mathfrak{g}}\tau$ [27–29], where $n_{\mathfrak{g}}$ is the ratio of the squared lengths of long and short roots. The transformations

$T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right): \tau \mapsto \tau + 1$ and $S = \left(\begin{smallmatrix}0 & -1/\sqrt{n_{\mathfrak{g}}}\\ \sqrt{n_{\mathfrak{g}}} & 0\end{smallmatrix}\right): \tau \mapsto -1/n_{\mathfrak{g}}\tau$ generate the Hecke group $G(\sqrt{n_{\mathfrak{g}}}) \subset \mathrm{SL}(2,\mathbb{R})$ and satisfy the relations

$$S^2 = (ST)^{2n_{\mathfrak{g}}} \in Z(G(\sqrt{n_{\mathfrak{g}}})), \quad S^4 = 1. \tag{4.1}$$

There are two sets of partition functions. In addition to $\{Z_u(\tau)\}_{u \in H^2(X,\mathcal{Z})}$ given by (3.1), we have $\{Z_\mu(\tau)\}_{\mu \in H^2(X,{}^L\mathcal{Z})}$, where

$$Z_\mu(\tau) = q^{-\check{s}} \sum_{k \in \mathbb{Z} - \frac{1}{2}(\mu|\mu)} \chi(\overline{\mathcal{M}_{k,\mu}})\, q^k \tag{4.2}$$

is the partition function of the theory with gauge group $({}^L G)_{\mathrm{ad}}$ and discrete flux $\mu \in H^2(X, {}^L\mathcal{Z})$. Here ${}^L\mathcal{Z} \cong {}^L\Lambda^*/{}^L\Lambda^\vee$ is the centre of $\widetilde{{}^L G}$. The generator $T: \tau \mapsto \tau+1$ transforms within each of the sets $\{Z_u(\tau)\}$ and $\{Z_\mu(\tau)\}$ while $S: \tau \mapsto -1/n_{\mathfrak{g}}\tau$ interchanges them.

To find how the Hecke group acts on the two sets of partition functions, we consider the theta functions on which the action of the Hecke group is known explicitly [30]. When G is non-simply laced, besides $\{\theta_a(\tau)\}_{a \in \mathcal{Z}}$ in (3.12), there is another set $\{\theta_\alpha(\tau)\}_{\alpha \in {}^L\mathcal{Z}}$, where

$$\theta_\alpha(\tau) = \sum_{\xi \in ({}^L\Lambda)^\vee + \alpha} e^{\pi\sqrt{-1}\,(\xi|\xi)}. \tag{4.3}$$

These theta functions are different from those in the affine characters [32, 33], which are sums over the lattice generated by the long roots and transform under the modular group. With (3.12) and (4.3), Poisson summation yields [30]

$$\begin{aligned}
\vartheta_a\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \frac{n_{\mathfrak{g}}^{r_{\mathrm{long}}/2}}{|\mathcal{Z}|^{1/2}} \left(\frac{\tau}{\sqrt{-1}}\right)^{r_{\mathfrak{g}}/2} \sum_{\alpha \in {}^L\mathcal{Z}} e^{-2\pi\sqrt{-1}\,\langle \alpha, a\rangle/\sqrt{n_{\mathfrak{g}}}}\, \vartheta_\alpha(z,\tau),\\
\vartheta_\alpha\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \frac{n_{\mathfrak{g}}^{r_{\mathrm{short}}/2}}{|\mathcal{Z}|^{1/2}} \left(\frac{\tau}{\sqrt{-1}}\right)^{r_{\mathfrak{g}}/2} \sum_{u \in \mathcal{Z}} e^{-2\pi\sqrt{-1}\,\langle \alpha, a\rangle/\sqrt{n_{\mathfrak{g}}}}\, \vartheta_a(z,\tau).
\end{aligned} \tag{4.4}$$

Thus we encounter the Hecke group. Let

$$\begin{aligned}
\hat{\vartheta}_a(\tau) &= \eta(\tau)^{-r_{\mathrm{long}}}\, \eta(n_{\mathfrak{g}}\tau)^{-r_{\mathrm{short}}}\, \vartheta_a(\tau),\\
\hat{\vartheta}_\alpha(\tau) &= \eta(\tau)^{-r_{\mathrm{short}}}\, \eta(n_{\mathfrak{g}}\tau)^{-r_{\mathrm{long}}}\, \vartheta_\alpha(\tau),
\end{aligned} \tag{4.5}$$

where r_{long} and r_{short} are the numbers of long, short simple roots of $\mathfrak{g}$, respectively. Then [30]

$$\begin{aligned}
\hat{\vartheta}_a(\tau+1) &= e^{-\pi\sqrt{-1}\, n_{\mathfrak{g}} r_{\mathfrak{g}} \check{h}({}^L\mathfrak{g})/12h(\mathfrak{g}) + \pi\sqrt{-1}\,(a|a)}\, \hat{\vartheta}_a(\tau),\\
\hat{\vartheta}_\alpha(\tau+1) &= e^{-\pi\sqrt{-1}\, n_{\mathfrak{g}} r_{\mathfrak{g}} \check{h}(\mathfrak{g})/12h(\mathfrak{g}) + \pi\sqrt{-1}\,(\alpha|\alpha)}\, \hat{\vartheta}_\alpha(\tau),\\
\hat{\vartheta}_a\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \frac{1}{|\mathcal{Z}|^{1/2}} \sum_{\alpha \in {}^L\mathcal{Z}} e^{-2\pi\sqrt{-1}\,\langle \alpha, a\rangle/\sqrt{n_{\mathfrak{g}}}}\, \hat{\vartheta}_\alpha(\tau),\\
\hat{\vartheta}_\alpha\left(-\frac{1}{n_{\mathfrak{g}}\tau}\right) &= \frac{1}{|\mathcal{Z}|^{1/2}} \sum_{a \in \mathcal{Z}} e^{-2\pi\sqrt{-1}\,\langle \alpha, a\rangle/\sqrt{n_{\mathfrak{g}}}}\, \hat{\vartheta}_a(\tau).
\end{aligned} \tag{4.6}$$

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The transformations of $Z_v(\tau)$ and $Z_\mu(\tau)$ under T are obvious. Following (3.2) and (4.4), we assume that the partition functions transform under S according to

$$\begin{aligned} Z_u\left(-\frac{1}{n_{\mathfrak g}\tau}\right) &= \frac{n_{\mathfrak g}^{w_{\rm long}/2}}{|\mathcal Z|^{b_2/2}}\left(\frac{\tau}{\sqrt{-1}}\right)^{w/2} \sum_{\mu\in H^2(X,{}^L\mathcal Z)} e^{2\pi\sqrt{-1}\,\langle\mu|u\rangle/\sqrt{n_{\mathfrak g}}}\, Z_\mu(\tau), \\ Z_\mu\left(-\frac{1}{n_{\mathfrak g}\tau}\right) &= \frac{n_{\mathfrak g}^{w_{\rm short}/2}}{|\mathcal Z|^{b_2/2}}\left(\frac{\tau}{\sqrt{-1}}\right)^{w/2} \sum_{u\in H^2(X,\mathcal Z)} e^{2\pi\sqrt{-1}\,\langle\mu|u\rangle/\sqrt{n_{\mathfrak g}}}\, Z_u(\tau), \end{aligned} \tag{4.7}$$

where $w = w_{\rm long} + w_{\rm short}$. Since S exchanges two sets of partition functions, there is no need for the $\pm$ sign that was present in (3.2). As in (3.3) and (4.5), let

$$\begin{aligned} \hat Z_u(\tau) &= \eta(\tau)^{-w_{\rm long}}\, \eta(n_{\mathfrak g}\tau)^{-w_{\rm short}}\, Z_u(\tau), \\ \hat Z_\mu(\tau) &= \eta(\tau)^{-w_{\rm short}}\, \eta(n_{\mathfrak g}\tau)^{-w_{\rm long}}\, Z_\mu(\tau). \end{aligned} \tag{4.8}$$

Then the transformations under T and S become

$$\hat Z_u(\tau+1) = e^{-\pi\sqrt{-1}\,c/12-\pi\sqrt{-1}\,(u|u)}\, \hat Z_u(\tau), \tag{4.9}$$

$$\hat Z_\mu(\tau+1) = e^{\pi\sqrt{-1}\,\check c/12-\pi\sqrt{-1}\,(\mu|\mu)}\, \hat Z_\mu(\tau), \tag{4.10}$$

$$\hat Z_u\left(-\frac{1}{n_{\mathfrak g}\tau}\right) = \frac{1}{|\mathcal Z|^{b_2/2}} \sum_{\mu\in H^2(X,{}^L\mathcal Z)} e^{2\pi\sqrt{-1}\,\langle\mu,u\rangle/\sqrt{n_{\mathfrak g}}}\, \hat Z_\mu(\tau), \tag{4.11}$$

$$\hat Z_\mu\left(-\frac{1}{n_{\mathfrak g}\tau}\right) = \frac{1}{|\mathcal Z|^{b_2/2}} \sum_{u\in H^2(X,\mathcal Z)} e^{2\pi\sqrt{-1}\,\langle\mu,u\rangle/\sqrt{n_{\mathfrak g}}}\, \hat Z_v(\tau), \tag{4.12}$$

where $c = 24s + w_{\rm long} + n_{\mathfrak g} w_{\rm short}$ and $\check c = 24\check s + w_{\rm short} + n_{\mathfrak g} w_{\rm long}$ are linear combinations of χ and σ. We have, just as (3.6),

$$\hat Z_G\left(-\frac{1}{n_{\mathfrak g}\tau}\right) = |Z(G)|^{-\chi/2}\,|Z({}^LG)|^{\chi/2}\,\hat Z_{{}^LG}(\tau), \tag{4.13}$$

recovering the original Montonen-Olive duality [1].

In [10], it was proposed that for non-simply laced groups, formula (3.3) holds if $c = c_1(\mathfrak g)\chi$, where $c_1(\mathfrak g) = \dim\mathfrak g/(1+\check h(\mathfrak g))$ is the central charge of the WZW model at level 1 [33]. We would like to suggest different values of c and $\check c$ so as to be compatible with the action of the Hecke group. We claim that (4.9) holds with

$$c = n_{\mathfrak g} r_{\mathfrak g} \frac{\check h({}^L\mathfrak g)}{h(\mathfrak g)}\chi = (r_{\rm long} + n_{\mathfrak g}\, r_{\rm short})\chi, \quad \check c = n_{\mathfrak g} r_{\mathfrak g} \frac{\check h(\mathfrak g)}{h(\mathfrak g)}\chi = (r_{\rm short} + n_{\mathfrak g}\, r_{\rm long})\chi. \tag{4.14}$$

As $\chi(\tilde X) = \chi(X)+1$, this is consistent with the factorisation

$$\hat Z_{\tilde X,\tilde u}(\tau) = \hat\theta_a(\tau)\,\hat Z_{X,u}(\tau), \quad \hat Z_{\tilde X,\tilde \mu}(\tau) = \hat\theta_\alpha(\tau)\,\hat Z_{X,\mu}(\tau), \tag{4.15}$$

where $\tilde u = (u, a\otimes e)$, $\tilde\mu = (\mu, \alpha\otimes e)$. The partitions functions of $\tilde X$ transform under the Hecke group in the tensor product representation of those of X and the theta functions.

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With the lack of mathematical examples in the non-simply laced case, it is not possible to fix the constants s, $\check{s}$, $w_{\rm long}$, $w_{\rm short}$ uniquely. A possible solution is

$$w_{\rm long}=-\frac{r_{\rm long}}{r_{\mathfrak g}}\chi,\quad w_{\rm short}=-\frac{r_{\rm short}}{r_{\mathfrak g}}\chi,\quad s=\frac{1+r_{\mathfrak g}^{-1}}{24}c,\quad \check{s}=\frac{1+r_{\mathfrak g}^{-1}}{24}\check{c}. \tag{4.16}$$

Then the blow-up formulae are

$$\begin{aligned}\sum_{k\in\mathbb{Z}-\frac12(\tilde u|\tilde u)}\chi(\,\overline{\mathcal{M}_{k,\tilde u}(\tilde X)}\,)\,q^k&=\left(\frac{q^{(r_{\rm long}+n_{\mathfrak g}r_{\rm short})/24}}{\eta(\tau)^{r_{\rm long}}\,\eta(n_{\mathfrak g}\tau)^{r_{\rm short}}}\right)^{1+r_{\mathfrak g}^{-1}}\theta_a(\tau)\sum_{k\in\mathbb{Z}-\frac12(u|u)}\chi(\,\overline{\mathcal{M}_{k,u}(X)}\,)\,q^k,\\ \sum_{k\in\mathbb{Z}-\frac12(\tilde\mu|\tilde\mu)}\chi(\,\overline{\mathcal{M}_{k,\tilde\mu}(\tilde X)}\,)\,q^k&=\left(\frac{q^{(r_{\rm short}+n_{\mathfrak g}r_{\rm long})/24}}{\eta(\tau)^{r_{\rm short}}\,\eta(n_{\mathfrak g}\tau)^{r_{\rm long}}}\right)^{1+r_{\mathfrak g}^{-1}}\theta_\alpha(\tau)\sum_{k\in\mathbb{Z}-\frac12(\mu|\mu)}\chi(\,\overline{\mathcal{M}_{k,\mu}(X)}\,)\,q^k.\end{aligned} \tag{4.17}$$

While the appearance of the theta functions matches [26], the fractional powers of the eta functions suggest that a more sophisticated compactification of the moduli spaces is necessary when the group is non-simply laced. It is possible to achieve integer powers at the expense of losing the symmetry between $\mathfrak{g}$ and ${}^L\mathfrak{g}$. (For example, the first factors on the right hand sides of the two equations in (4.17) can be replaced by $q^{(r_{\mathfrak g}+1)/24}/\eta(\tau)^{r_{\mathfrak g}+1}$ and $q^{n_{\mathfrak g}(r_{\mathfrak g}+1)/24}/\eta(n_{\mathfrak g}\tau)^{r_{\mathfrak g}+1}$, respectively.) Then the moduli spaces $\mathcal{M}_{k,u}$ and $\mathcal{M}_{k,\mu}$ would have to be compactified differently.

We check that with the choices of c and $\check{c}$ in (4.14), the matrices

$$\begin{aligned}&\mathcal{T}_{uv}=e^{-\pi\sqrt{-1}\,c/12-\pi\sqrt{-1}\,(v|v)}\,\delta_{uv},\qquad \check{\mathcal{T}}_{\mu\nu}=e^{-\pi\sqrt{-1}\,\check{c}/12-\pi\sqrt{-1}\,(\mu|\mu)}\,\delta_{\mu\nu},\\ &\mathcal{S}_{\mu u}=|\mathcal{Z}|^{-b_2/2}\,e^{2\pi\sqrt{-1}\,\langle\mu,v\rangle/\sqrt{n_{\mathfrak g}}}=\check{\mathcal{S}}_{u\mu}\end{aligned} \tag{4.18}$$

($u,v\in H^2(X,\mathcal{Z})$, $\mu,\nu\in H^2(X,{}^L\mathcal{Z})$) from (4.9) indeed satisfy the relations in (4.1) for the Hecke group. For non-simply laced simple Lie algebras, the centre $\mathcal{Z}$ is either $\mathbb{Z}_2$ (for B_r and C_r) or trivial (for F_4 and G_2). Therefore all u and μ are two-torsions and $\mathcal{S}^2$ and $\check{\mathcal{S}}^2$ are the identity matrix. We show that $(\check{\mathcal{S}}\check{\mathcal{T}}\mathcal{S}\mathcal{T})^{n_{\mathfrak g}}$ is also the identity matrix.

First, the contribution of new phase factors in (4.9) involving c and $\check{c}$ is

$$(e^{-\pi\sqrt{-1}\,c/12}e^{-\pi\sqrt{-1}\,\check{c}/12})^{n_{\mathfrak g}}=e^{-\pi\sqrt{-1}\,n_{\mathfrak g}(n_{\mathfrak g}+1)r_{\mathfrak g}\,\chi/12} \tag{4.19}$$

by using the second identity in (A.4). If $\mathfrak{g}$ is of type F_4 or G_2, then $\mathcal{Z}=\{1\}$ and all the matrices concerned are scalars. It is easy to check that (4.19) is equal to 1 in both cases. (If $c=c_1(\mathfrak{g})\chi$, then (4.19) would be $e^{4\pi\sqrt{-1}\,\chi/15}$ for F_4 and $e^{3\pi\sqrt{-1}\,\chi/5}$ for G_2.) In this case, each of the two sets of partition functions contain one element, $Z_{u=0}$ and $Z_{\mu=0}$, respectively, which transform under the Hecke group. Since the Langlands dual group is isomorphic to the original group, the two partition functions are equal unless the compactifications $\overline{\mathcal{M}_{k,u=0}}$ and $\overline{\mathcal{M}_{k,\mu=0}}$ are different (a possibility suggested above).

If $\mathfrak{g}$ is of type B_r or C_r, say C_r, then the discrete fluxes are $u=x\otimes\check{\lambda}_s$ and $\mu=y\otimes\check{\lambda}_1$ for some $x,y\in H^2(X,\mathbb{Z}_2)$. Here $\check{\lambda}_s$ and $\check{\lambda}_1$ are, respectively, the fundamental coweights corresponding to the spinor representation of B_r and the defining representation of C_r. (Both representations are miniscule.) A straightforward calculation yields

$$(u|u)=r\,x^2/2,\quad (\mu|\mu)=y^2,\quad \langle\mu,u\rangle/\sqrt{2}=x\cdot y\mod 2, \tag{4.20}$$

where the pairing $x \cdot y$ is explained in appendix B. Using the Wu formula (B.5), we have

$$\begin{aligned}(\check{\mathcal{S}}\check{\mathcal{T}}\mathcal{S}\mathcal{T})_{xy} &= e^{-\pi\sqrt{-1}\,r\chi/4}\,2^{-b_2} \sum_{z\in H^2(X,\mathbb{Z}_2)} e^{\pi\sqrt{-1}\,x\cdot z} e^{-\pi\sqrt{-1}\,z^2} e^{\pi\sqrt{-1}\,z\cdot y} e^{-\pi\sqrt{-1}\,r\,y^2/2} \\ &= e^{-\pi\sqrt{-1}\,(\chi/4+r\,y^2/2)}\,\delta_{x+y,w_2} \end{aligned} \tag{4.21}$$

and

$$((\check{\mathcal{S}}\check{\mathcal{T}}\mathcal{S}\mathcal{T})^2)_{xy} = e^{-\pi\sqrt{-1}\,r(\chi+w_2^2)/2}\,\delta_{xy}. \tag{4.22}$$

The phase factor on the right hand side is 1 by (B.6) and (B.8).

5. Conclusions

When the gauge group is non-simply laced, we proposed how the partition functions of the Vafa-Witten theory transform under the generators of the Hecke group. Our approach was to use the known transformation of the theta functions and their appearance in the blow-up formulae. As a consistency check, we verified that these transformations indeed define a representation of the Hecke group on two sets of partition functions for the gauge group and its Langlands dual. However, much remains to be done. One of the important problems is to compute the partition functions, either mathematically for simple examples of four-manifolds such as K3 and rational surfaces or by mass deformation for Kähler surfaces whose canonical divisor is a disjoint union of smooth curves. Another is to clarify the meaning of fractional powers of the eta functions in the blow-up formulae using the appropriate compactification of moduli spaces. We leave these questions for future exploration.

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A. Notations and facts of Lie groups and Lie algebras

In this paper, G is a simple, connected, compact Lie group with Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Then $T = \mathfrak{t}/2\pi\sqrt{-1}\,\ell$ for some lattice $\ell \subset \sqrt{-1}\,\mathfrak{t}$. Let $\Delta \subset \sqrt{-1}\,\mathfrak{t}^*$ be the root system and $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\} \subset \sqrt{-1}\,\mathfrak{t}$, the coroot system. Denote by Λ and $\Lambda^\vee$ the root and coroot lattices, respectively. Then the weight and coweight lattices are $(\Lambda^\vee)^*$ and Λ^*. We have the inclusions [34], §IX.4.9

$$\Lambda^\vee \subset \ell \subset \Lambda^* \subset \sqrt{-1}\,\mathfrak{t}, \quad \Lambda \subset \ell^* \subset (\Lambda^\vee)^* \subset \sqrt{-1}\,\mathfrak{t}^*. \tag{A.1}$$

Let $\tilde{G}$ be the universal covering group of G. Its centre is $\mathcal{Z} \cong \Lambda^*/\Lambda^\vee$. The adjoint group $G_{\mathrm{ad}} = \tilde{G}/\mathcal{Z}$ has $\pi_1(G_{\mathrm{ad}}) = \mathcal{Z}$ and a trivial centre. In general, $\pi_1(G) \cong \ell/\Lambda^\vee$ and

$Z(G) \cong \Lambda^*/\ell$. The maximal tori of $\tilde{G}$ and $G_{\rm ad}$ are, respectively, $\tilde{T} = \mathfrak{t}/2\pi\sqrt{-1}\,\Lambda^\vee$ and $T_{\rm ad} = \mathfrak{t}/2\pi\sqrt{-1}\,\Lambda^*$.

We fix an inner product $(\cdot|\cdot)$ on $\sqrt{-1}\,\mathfrak{g}$ such that the long roots are of square length 2. Let $n_\mathfrak{g}$ be the ratio of square lengths of long and short roots. $\mathfrak{g}$ is simply laced if all roots are of the same length, in which case we set $n_\mathfrak{g} = 1$. Otherwise, $\mathfrak{g}$ is non-simply laced and $n_\mathfrak{g}$ is either 2 or 3. The Langlands dual ${}^L\mathfrak{g}$ of $\mathfrak{g}$ is the Lie algebra whose root system is isomorphic to $\Delta^\vee$. To keep the same normalisation on the inner product, the root system of ${}^L\mathfrak{g}$ should be ${}^L\Delta = n_\mathfrak{g}^{-1/2}\Delta^\vee$. Thus its (co)root and (co)weight lattices are

$$ {}^L\Lambda = n_\mathfrak{g}^{-1/2}\Lambda^\vee, \quad ({}^L\Lambda)^\vee = n_\mathfrak{g}^{1/2}\,\Lambda, \quad (({}^L\Lambda)^\vee)^* = n_\mathfrak{g}^{-1/2}\Lambda^*, \quad ({}^L\Lambda)^* = n_\mathfrak{g}^{1/2}\,(\Lambda^\vee)^*. \tag{A.2} $$

The Lie algebra ${}^L\mathfrak{g}$ determines a simply connected Lie group $\widetilde{{}^LG}$ whose centre is ${}^L\mathcal{Z} \cong ({}^L\Lambda)^*/({}^L\Lambda)^\vee \cong (\Lambda^\vee)^*/\Lambda$ and is isomorphic to $\mathcal{Z}^\wedge = \mathrm{Hom}(\mathcal{Z}, U(1))$, the character group of $\mathcal{Z}$. The Langlands dual ${}^LG$ of the group G is defined by specifying $\pi_1({}^LG)$ as the subgroup of characters on $\mathcal{Z}$ that is trivial on $\pi_1(G)$. We have $\pi_1({}^LG) \cong Z(G)^\wedge$ and $Z({}^LG) = \pi_1(G)^\wedge$. In particular, ${}^L\tilde{G} = ({}^LG)_{\rm ad}$ and ${}^L(G_{\rm ad}) = \widetilde{{}^LG}$.

The centre $\mathcal{Z}$ is closely related to the miniscule representations of ${}^L\mathfrak{g}$. A representation of $\mathfrak{g}$ is miniscule if the weights form a single orbit under the Weyl group action. If so, the highest weight is called a miniscule weight. A miniscule weight is fundamental, but not conversely. The miniscule weights are in one-to-one correspondence with the non-zero elements of $(\Lambda^\vee)^*/\Lambda$, by sending the weight to the coset it represents [34], §VIII.7.3. Thus there is a bijection between the set of miniscule and zero weights and ${}^L\mathcal{Z}$ [30]. A representation of G is miniscule if the induced representation of $\mathfrak{g}$ is so. There is a bijection between the set of miniscule and trivial representations of G and $\pi_1({}^LG) \cong Z(G)^\wedge$ [30].

We mention some results related to the normalised inner product $(\cdot|\cdot)$. First, the Killing form $\kappa(\cdot,\cdot)$, extended complex linearly to $\mathfrak{g}^\mathbb{C}$ and restricted to $\sqrt{-1}\,\mathfrak{t}$, is positive definite. We have $\kappa(x,y) = 2\check{h}(\mathfrak{g})(x|y)$ for any $x, y \in \sqrt{-1}\,\mathfrak{t}$, where $\check{h}(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$. We recall that the Coxeter number $h(\mathfrak{g})$ of $\mathfrak{g}$ satisfies $|\Delta| = r_\mathfrak{g}\,h(\mathfrak{g})$, where $r_\mathfrak{g}$ is the rank of $\mathfrak{g}$ [34], §V.6.2. Second, we have for any $\alpha \in \Delta$, $y \in \Lambda^*$, $(\check{\alpha}|\check{\alpha}) = 4/(\alpha|\alpha) \in 2\mathbb{Z}$ and $(\check{\alpha}|y) = 2\langle\alpha, y\rangle/(\alpha|\alpha) \in \mathbb{Z}$. Consequently, for any $x \in \Lambda^\vee$, $y \in \Lambda^*$, we have

$$ (x|x) \in 2\mathbb{Z}, \quad (x|y) \in \mathbb{Z}. \tag{A.3} $$

However, when both $x, y \in \Lambda^*$, $(x|y) \in \mathbb{Q}$ is not always an integer. We denote by $m_\mathfrak{g}$ the smallest positive integer m so that $m(x|x)/2 \in \mathbb{Z}$ for all $x \in \Lambda^*$. Since $\Lambda^*/\Lambda^\vee$ is of order $|\mathcal{Z}|$, $m_\mathfrak{g}$ divides $2|\mathcal{Z}|$. We list $m_\mathfrak{g}$ together with other data for each type of $\mathfrak{g}$ in table 1.

Finally, we relate the Coxeter and dual Coxeter numbers of $\mathfrak{g}$ to those of ${}^L\mathfrak{g}$. If $\mathfrak{g}$ is simply laced, then ${}^L\mathfrak{g} \cong \mathfrak{g}$ and $\check{h}(\mathfrak{g}) = h(\mathfrak{g})$. In general, we have [30]

$$ h({}^L\mathfrak{g}) = h(\mathfrak{g}), \quad \check{h}(\mathfrak{g}) + \check{h}({}^L\mathfrak{g}) = (1 + n_\mathfrak{g}^{-1})\,h(\mathfrak{g}). \tag{A.4} $$

The first identity follows easily from $|{}^L\Delta| = |\Delta|$ while the second, from (see for example [30])

$$ \check{h}(\mathfrak{g}) = (r_{\rm long} + n_\mathfrak{g}^{-1} r_{\rm short})h(\mathfrak{g})/r_\mathfrak{g} \tag{A.5} $$

$\mathfrak{g}$	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$n_\mathfrak{g}$	1	2	2	1	1	1	1	2	3
$m_\mathfrak{g}$	$2(r+1)/(2,r)$	1	$2/(2,r)$	$8/(4,r)$	3	4	1	1	1
$h(\mathfrak{g})$	$r+1$	$2r$	$2r$	$r+1$	12	18	30	12	6
$\check{h}(\mathfrak{g})$	$r+1$	$2r-1$	$r+1$	$2r-2$	12	18	30	9	4

Table 1: Important data of simple Lie algebras. Here (m,n) denotes the greatest common divisor of two positive integers m and n.

and the same equality for ${}^L\mathfrak{g}$. Here $r_{\rm long}$ and $r_{\rm short}$ are the numbers of long, short simple roots of $\mathfrak{g}$, respectively. We give a simple proof of (A.5). Using $\sum_{\gamma\in\Delta}\kappa_{\alpha\gamma}\kappa_{\gamma\beta}=\kappa_{\alpha\beta}$, where $\kappa_{\alpha\beta}=\kappa(\alpha,\beta)$ for $\alpha,\beta\in\Delta$, the trace of the matrix $(\kappa_{\alpha\beta})_{\alpha,\beta\in\Delta}$ is $\sum_{\alpha\in\Delta}\kappa_{\alpha\alpha}=r_\mathfrak{g}$ [35]. This implies that

$$|\Delta_{\rm long}|+n_\mathfrak{g}^{-1}|\Delta_{\rm short}|=r_\mathfrak{g}\,\check{h}(\mathfrak{g}), \tag{A.6}$$

where $\Delta_{\rm long}$ and $\Delta_{\rm short}$ are the sets of long and short roots, respectively. Since

$$|\Delta_{\rm long}|=r_{\rm long}\,h(\mathfrak{g}), \quad |\Delta_{\rm short}|=r_{\rm short}\,h(\mathfrak{g}) \tag{A.7}$$

(see [34], exer. VI.1.20), the result follows.

B. The geometry of instanton numbers and discrete fluxes

Topologically, principal G-bundles P over a compact, orientable, smooth four-manifold X are classified by $p_1({\rm ad}P)\in H^4(X,\mathbb{Z})$ and $w_2(P)=w_2({\rm ad}P)\in H^2(X,\pi_1(G))$. The former determines the instanton number

$$k(P)=-\langle p_1({\rm ad}P),[X]\rangle/2\check{h}(\mathfrak{g})\in\mathbb{Q} \tag{B.1}$$

and the latter is a discrete flux [11] and is the obstruction to lift P to a $\tilde{G}$-bundle. If G itself is simply connected, then $k(P)\in\mathbb{Z}$ and it is the only characteristic number of P. To get the most general $w_2(P)$, we take $G=G_{\rm ad}$. Since $\pi_1(G_{\rm ad})=\mathcal{Z}$, we have a long exact sequence

$$\cdots\to H^2(X,\Lambda^\vee)\to H^2(X,\Lambda^*)\to H^2(X,\mathcal{Z})\to H^3(X,\Lambda^\vee)\to\cdots. \tag{B.2}$$

We assume that all elements in $H^2(X,\mathcal{Z})$ can be lifted to $H^2(X,\Lambda^*)$. If $\tilde{w}\in H^2(X,\Lambda^*)$ is a lift of $w_2(P)\in H^2(X,\mathcal{Z})$, then there is a $T_{\rm ad}$-bundle $Q\to X$ whose first Chern class is $c_1(Q)=\tilde{w}$. We denote by Q^{-1} the $T_{\rm ad}$-bundle with $c_1(Q^{-1})=-\tilde{w}$. The bundles P and Q^{-1} are both quotients of a $\tilde{G}\times_\mathcal{Z}\tilde{T}$-bundle over X constructed as follows. Let $\{U_\alpha\}$ be a good open cover of X and let $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to G_{\rm ad}$ be the transition functions of P. If we lift $g_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\tilde{G}$, then the functions $h_{\alpha\beta\gamma}=\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}\colon U_\alpha\cap U_\beta\cap U_\gamma\to\mathcal{Z}$ form a Čech cocycle that represents $w_2(P)$. Let $t_{\alpha\beta}\colon U_\alpha\cap U_\beta\to T_{\rm ad}$ be the transition functions of Q. The fact that $c_1(Q)=\tilde{w}$ is a lift of $w_2(P)$ means that $t_{\alpha\beta}$ can be lifted to $\tilde{t}_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\tilde{T}$ so that $\tilde{t}_{\alpha\beta}\tilde{t}_{\beta\gamma}\tilde{t}_{\gamma\alpha}=h_{\alpha\beta\gamma}$ on $U_\alpha\cap U_\beta\cap U_\gamma$. The $\tilde{G}\times_\mathcal{Z}\tilde{T}$-bundle is

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defined by the transition functions $(g_{\alpha\beta}, t_{\alpha\beta}^{-1})$ modulo the diagonal $\mathcal{Z}$-action on $\tilde{G} \times \tilde{T}$. If $G = SO(n)$, then $\tilde{G} \times_{\mathcal{Z}} \tilde{T} = \mathrm{Spin}^{\mathbb{C}}(n)$ and the existence of a lift $\tilde{w}$ is equivalent to that of a spin$^{\mathbb{C}}$ structure on P.

Consider now the associated bundle $\mathrm{ad}'Q = Q \times_{T_{\mathrm{ad}}} \mathfrak{g}$, where T_{ad} acts on $\mathfrak{g}$ by the adjoint representation. The two vector bundles $\mathrm{ad}'Q$ and $\mathrm{ad}P = P \times_{G_{\mathrm{ad}}} \mathfrak{g}$ have the same fibre $\mathfrak{g}$ and the same $w_2(\mathrm{ad}'Q) = w_2(\mathrm{ad}P)$. Consequently, they are isomorphic outside one point in X and their instanton numbers differ by that of a bundle on S^4, which is an integer [10]. To calculate the instanton number of $\mathrm{ad}'Q$, we note that

$$p_1(\mathrm{ad}'Q) = \frac{1}{2}\,\mathrm{ch}_2(\mathrm{ad}'Q) = \frac{1}{2}\sum_{\alpha\in\Delta} \langle \alpha, \tilde{w}\rangle^2 = \check{h}(\mathfrak{g})\,(\tilde{w}|\tilde{w}). \tag{B.3}$$

Here $(\cdot|\cdot)$ on $H^2(X, \Lambda^*)$ is defined by the normalised inner product $(\cdot|\cdot)$ on $\sqrt{-1}\,\mathfrak{t}$ and the intersection form on $H^2(X,\mathbb{Z})$. Thus we have $k(\mathrm{ad}'Q) = -\frac{1}{2}(\tilde{w}|\tilde{w})$. The number $\frac{1}{2}(\tilde{w}|\tilde{w})$ mod 1 is independent on the lift $\tilde{w}$ of $w_2(P)$. In fact, if $\tilde{w}'$ is another lift, then $\tilde{w}' - \tilde{w} \in H^2(X, \Lambda^\vee)$ and $\frac{1}{2}(\tilde{w}'|\tilde{w}') - \frac{1}{2}(\tilde{w}|\tilde{w}) = (\tilde{w}|\tilde{w}' - \tilde{w}) + \frac{1}{2}(\tilde{w}' - \tilde{w}|\tilde{w}' - \tilde{w}) \in \mathbb{Z}$ by (A.3). So we have

$$k(P) = -\frac{1}{2}\,(w_2(P)|w_2(P)) \bmod 1. \tag{B.4}$$

An important consequence of (B.4) is that for G_{ad}-bundles, instanton numbers are not necessarily integers [10]. For $G_{\mathrm{ad}} = \mathrm{SU}(n)/\mathbb{Z}_n$ and $w_2(P) = x \otimes \check{\lambda}_1$, where $x \in H^2(X, \mathbb{Z}_n)$ and $\check{\lambda}_1$ is the fundamental (and miniscule) coweight of $\mathfrak{g}$ that corresponds to the defining representation, we have $\frac{1}{2}(w_2(P)|w_2(P)) = x^2(n{-}1)/2n$ mod 1, which is (3.9) (when $n = 2$) and (3.13) of [10]. Here $x^2 = \langle x \cup x, [X]\rangle$ and $x^2(n-1)/2n$ is well-defined modulo 1. The generalisation to arbitrary simply laced groups is straightforward [10]. With the proper normalisation $(\cdot|\cdot)$ on $\sqrt{-1}\,\mathfrak{t}$, we have (B.4) for non-simply laced Lie groups as well. Since it is possible to glue instantons [36] on S^4 to X without affecting w_2, one can exhaust all numbers satisfying (B.4) by choosing various G_{ad}-bundles with a fixed w_2. However, a non-trivial $w_2(P)$ is not always reflected by a fractional instanton number. For each type of simple Lie algebra $\mathfrak{g}$, the number $m_{\mathfrak{g}}$ in appendix A is the smallest positive integer m such that $m\,k(P)$ is always an integer. It can be improved to $m_{\mathfrak{g}}/2$ if X is spin and $m_{\mathfrak{g}}$ is even.

We consider the tangent bundle TX. The second Stiefel-Whitney class $w_2 = w_2(X) = w_2(TX) \in H^2(X, \mathbb{Z}_2)$ always lifts to $H^2(X,\mathbb{Z})$. This means that X is always spin$^{\mathbb{C}}$; X is spin if and only if $w_2 = 0$. The Wu formula is

$$x^2 = x \cdot w_2 \bmod 2, \tag{B.5}$$

where for $x, y \in H^2(X, \mathbb{Z}_2)$, $x \cdot y = \langle x \cup y, [X]\rangle$ is defined modulo 2. If x lifts to $H^2(X,\mathbb{Z})$, then $x^2 = x \cdot x$ is defined modulo 4. Moreover, w_2^2 is defined modulo 8. It is a classical result [37, 38] (see [39, 40] for further developments) that the signature $\sigma = \sigma(X)$ of X satisfies

$$\sigma = w_2^2 \bmod 8. \tag{B.6}$$

Finally, let $\mathcal{M}_{k,v} = \mathcal{M}_{k,v}(X)$ be the moduli space of anti-self-dual connections on a G-bundle $P \to X$ with instanton number k and discrete flux v. Its dimension is [41]

$$\dim \mathcal{M}_{k,v} = -2\,\langle p_1(\mathrm{ad}P), [X]\rangle - \frac{1}{2}\dim G\;(\chi + \sigma), \tag{B.7}$$

where $\chi = \chi(X)$ is the Euler number of X. Since $\chi + \sigma$ is even for any four-manifold and since $\dim G = r_{\mathfrak{g}}$ mod 2, the dimension (B.7) is even if and only if

$$r_{\mathfrak{g}}\,(\chi + \sigma) = 0 \bmod 4. \tag{B.8}$$

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